

Derive Boltzmann Equation for Cold Dark matter:

$$\frac{df}{dt} = \underbrace{\frac{\partial f}{\partial t}}_{\textcircled{1}} + \underbrace{\frac{\partial f}{\partial x^i} \frac{dx^i}{dt}}_{\textcircled{2}} + \underbrace{\frac{\partial f}{\partial p} \frac{dp}{dt}}_{\textcircled{3}} + \underbrace{\frac{\partial f}{\partial p^i} \frac{dp^i}{dt}}_{\textcircled{4}} = 0 \quad \langle 0 \rangle$$

For CDM, $\frac{\partial f}{\partial p^i} = 0$, so $\textcircled{4}$ vanishes.

For term $\textcircled{2}$, we must calculate $\frac{dx^i}{dt} = \frac{1}{p^0} \frac{dx^i}{d\lambda} = \frac{p^i}{p^0}$

Let us assume $p^2 = g_{ij} p^i p^j \quad \langle 1 \rangle$

$$\left\{ \begin{array}{l} -m^2 = g_{\mu\nu} p^\mu p^\nu \quad \langle 2 \rangle \\ p^i = c \hat{p}^i \quad \langle 3 \rangle \\ \delta_{ij} \hat{p}^i \hat{p}^j = 1 \quad \langle 4 \rangle \\ E^2 = p^2 + m^2 \quad \langle 5 \rangle \end{array} \right.$$

From $\langle 1 \rangle \langle 2 \rangle \langle 5 \rangle$, we obtain $p^0 = E(1-\Psi)$

From $\langle 1 \rangle \langle 3 \rangle \langle 4 \rangle$, we obtain $p^i = \frac{\hat{p}^i}{a} (1-\Phi)$

Now we have $\frac{dx^i}{dt} = \frac{p^i}{p^0} = \frac{\hat{p}^i}{aE} (1-\Phi + \Psi)$

$$\textcircled{2} = \frac{\partial f}{\partial x^i} \frac{dx^i}{dt} \stackrel{\text{keep to 1st order}}{=} \frac{\partial f}{\partial x^i} \frac{\hat{p}^i}{aE}$$

For term $\textcircled{3}$, we need to derive $\frac{dp}{dt}$.

From geodesic equation $\frac{dp^\alpha}{d\lambda} = -\Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma$

$$-r.h.s = \frac{1}{2} g^{\alpha\mu} (\partial_\alpha g_{\mu\rho} + \partial_\rho g_{\mu\alpha} - \partial_\mu g_{\alpha\rho}) p^\rho p^\beta$$

$$= -\frac{1}{2} (1-2\Psi) \left(-4 \partial_\alpha \Psi p^\alpha p^0 - \frac{\partial}{\partial t} g_{\alpha\beta} p^\alpha p^\beta \right)$$

$$= -\frac{1}{2} (1-2\Psi) \left(-4 \frac{\partial \Psi}{\partial t} E^2 (1-\Phi)^2 - 4 \frac{\partial \Psi}{\partial x^i} \frac{E \hat{p}^i}{a} (1-\Psi-\Phi) + \frac{\partial}{\partial t} (1+2\Psi) \cdot E^2 (1-\Psi)^2 \right)$$

keep to 1st order

$$\stackrel{=} {=} p^2 \left(\frac{\partial \Psi}{\partial t} \frac{E^2}{p^2} + 2 \frac{E \hat{p}^i}{pa} \frac{\partial \Psi}{\partial x^i} + \dot{\Phi} + H - 2H\Psi \right)$$

$$\begin{aligned} l.h.s &= \frac{dp^0}{dt} = p^0 \frac{dp^0}{dt} = E(1-\Psi) \frac{d}{dt} (E(1-\Psi)) \\ &= -E^2(1-\Psi) \left(\frac{\partial \Psi}{\partial t} + \frac{\partial \Psi}{\partial x^i} \frac{dx^i}{dt} + E(1-\Psi)^2 \frac{dE}{dt} \right) \end{aligned}$$

use $p \cdot p = E^2$

keep to 1st order

$$\stackrel{=} {=} p \frac{dp}{dt} (1-2\Psi) - E^2 \frac{\partial \Psi}{\partial t} - E^2 \frac{\partial \Psi}{\partial x^i} \frac{\hat{p}^i}{aE}$$

Now let $l.h.s = r.h.s$, we obtain

$$\frac{dp}{dt} \stackrel{\text{to 1st order}}{=} -p \left(\dot{\Phi} + H + \frac{\partial \Psi}{\partial x^i} \frac{E \hat{p}^i}{pa} \right)$$

Substitute $\textcircled{1} \sim \textcircled{4}$ into $\langle 0 \rangle$, we have

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{aE} \frac{\partial f}{\partial x^i} - \frac{\partial f}{\partial E} \left(H \frac{p^2}{E} + \frac{\partial \Phi}{\partial t} \frac{p^2}{E} + \frac{\hat{p}^i}{a} \frac{\partial \Psi}{\partial x^i} \right) \quad \langle 7 \rangle$$

Then we define $n(t, \vec{x}) = \int \frac{d^3 p}{(2\pi)^3} f$

$$v^i(t, \vec{x}) = \int \frac{d^3 p}{(2\pi)^3} f \frac{\hat{p}^i}{E} \cdot \frac{1}{n}$$

$\int \frac{d^3 p}{(2\pi)^3} \times \langle 7 \rangle$, in class we have shown the 0-momentum equation to be ($n = n_0(t) (1 + \delta(t, \vec{x}))$)

$$\frac{\partial n}{\partial t} + \frac{1}{a} \frac{\partial}{\partial x^i} (n v^i) + 3n \left(H + \frac{\partial \Phi}{\partial t} \right) \quad \langle 8 \rangle$$

Expand $\langle 8 \rangle$ to 0-th order, we obtain

$$\frac{\partial n^{(0)}}{\partial t} + 3n^{(0)} H = 0 \Rightarrow n^{(0)} \propto a^{-3} \quad \langle 9 \rangle$$

Expand $\langle 8 \rangle$ to 1-st order, we obtain (use $\langle 9 \rangle$)

$$\frac{\partial \delta}{\partial t} + \frac{1}{a} \frac{\partial}{\partial x^i} v^i + 3 \frac{\partial \Phi}{\partial t} = 0$$

change t to conformal time η

$$\Rightarrow \frac{\partial \delta}{\partial \eta} + \frac{\partial v^i}{\partial x^i} + 3 \frac{\partial \Phi}{\partial \eta} = 0$$

transform to Fourier space

$$\Rightarrow \dot{\delta} + i\vec{k} \cdot \vec{v} + 3\dot{\Phi} = 0 \quad \langle 10 \rangle$$

$\vec{k} \parallel \vec{v}$ in linear theory, $i\vec{k} \cdot \vec{v} = ik\tilde{v}$

Now $\int \frac{d^3 p}{(2\pi)^3} \frac{\hat{p}^i}{E} \times \langle 7 \rangle$, we obtain the 1-momentum equation in class

$$\frac{\partial (n v^i)}{\partial t} + 4H n v^i + \frac{n}{a} \frac{\partial \Psi}{\partial x^i} = 0 \quad \langle 11 \rangle$$

Expand $\langle u \rangle$ to 1th order, we obtain (use $\langle q \rangle$)

$$\frac{\partial v^i}{\partial t} + H v^i + \frac{1}{a} \frac{\partial \Phi}{\partial x^i} = 0$$

change t
to conformal
time η

$$\Rightarrow \frac{\partial v^i}{\partial \eta} + \frac{1}{a} \frac{\partial \Phi}{\partial x^i} = 0$$

transform to
Fourier space

$$\Rightarrow \dot{\tilde{v}}^i + \frac{1}{a} \frac{\partial \tilde{\Phi}}{\partial t} + ik^i \tilde{\Phi} = 0 \quad \langle 12 \rangle$$