Generalization of Sturm-Liouville Theorem to Dirac Equation

It goes without saying that Sturm-Liouville theory of ordinary differential equations of second order has been successfully applied to physics. In the case of non-relativistic quantum mechanics, the state or movement of a microscopic particle obeys Schrödinger equation. If the potential is independent of time, the time factor of the wave function can be separated, and the space coordinate factor satisfies the stationary Schrödinger equation. This is an ordinary differential equation of second order, and its eigenvalues are the energies the system may have. The corresponding eigenfunctions are the states the system may exists. The third Sturm-Liouville theorem tells us that the eigenenergy has a lower limit. In a quantum system, the energy has indeed a minimum. The state corresponding to the minimum energy is called ground state. The states with higher energies are called excitation states. When the system is not disturbed, the particle is in its ground state. When an external field is applied, the particle may transit from the ground state to an excitation state. In the course of transition, the total energy is conserved.

However, in the case of relativistic quantum mechanics, new features appear. A relativistic particle obeys Dirac equation of relativistic quantum mechanics. Solving Dirac equation, one obtains the energy expressed by $E = \pm c\sqrt{p^2 + m^2c^2}$, where m and p are the particle's rest mass and momentum, respectively, and c is light speed. It is seen that there are two branches of energies: positive and negative ones. The positive branch has a lower limit but not the upper limit, which agrees with the Sturm-Liouville theory. However, the negative branch has no lower limit, which is not consistent with the Sturm-Liouville theory. The appearance of the branch without a lower limit arised from the fact that the form of Dirac equation is different from that of Sturm-Liouville equations. Therefore, it was desirable to extend the Sturm-Liouville theory to one that could be applied to Dirac equation. C. N. Yang did this work (Yang C N，Generalization of Sturm-Liouville Theory to a System of Ordinary Differential Equations with Dirac Type Spectrum. Commun. Math. Phys., 1987, **112**: 205-216). He noticed that in Dirac equation, derivations were written in a matrix form. Consequently, the solution functions had to comprise at least two components. He proposed four theorems applicable to Dirac equation. His theory was parallel to Sturm-Liouville theory. Here we do not intend to introduce his theory. Readers are encouraged to read his article.